PERIODIC ELECTROSTATIC FOCUSING OF A BEAM OF ELLIPTICAL CROSS SECTION

V. A. Syrovoi

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ABSTRACT: An exact analytic solution has been given [1] for the shaping of a flat beam with a nonmonotonic potential variation at the boundary. It has been shown that construction of a periodic focusing system amounts to calculation of the equipotentials in a part with a potential distribution symmetric about the center and to coupling of two such elements via sufficiently thick screening grids whose charge density varies in a definite way. Here I derive a simple analytic solution for the shaping of a beam of elliptical cross section by approximating the potential at the boundary via a quadratic parabola [2]. The result for high eccentricity may serve as a model for edge effects in flat beams of finite width.

We use the elliptical cylindrical coordinates ξ, η, z of Fig. 1, which are related to the Cartesian coordinates x, y, z by

$$x = a\sqrt{\beta-1}\,\mathrm{sh}\,\xi\sin\eta, \qquad y = a\sqrt{\beta-1}\,\mathrm{ch}\,\xi\cos\eta, \qquad z = z. \tag{1}$$

Let $\xi_0 \le \xi < \infty$, $0 \le \eta \le 2\pi$ be the Laplace region of the boundary ellipse $\xi = \xi_0$, $\beta = (b/a)^2$ whose semiminor axis is a. The problem is to solve

$$\partial^2\varphi/\partial\xi^2 + \partial^2\varphi/\partial\eta^2 + a^2(\beta-1)\times$$
$$\times(\mathrm{sh}^2\,\xi + \sin^2\eta)\,\partial^2\varphi/\partial z^2 = 0, \tag{2}$$

which satisfies the following conditions at the boundary $\xi = \xi_0$:

$$\varphi = \varphi_0(z) = \alpha + (1-\alpha),$$
$$(z/\sigma - 1)^2 = \alpha + (1-\alpha)\,Z^2,$$
$$\partial\varphi/\partial\xi = \varphi_1 = 0. \tag{3}$$

Here α is the minimum potential $\alpha \le \varphi \le 1$ in the range $0 \le z \le 2\sigma$, $\sigma = (1 + 2\alpha^{1/2})(1-\alpha)^{1/2}$. Dimensionless variables [1] will be used to find the solution as a series in $(\xi - \xi_0)$ with coefficients dependent on η and z:

$$\varphi = \varphi_k(\xi-\xi_0)^k \qquad (k = 0,1,\ldots), \tag{4}$$

for which purpose the expression before $\partial^2\varphi/\partial z^2$ in (2) is put in analogous form:

$$a^2(\beta-1)(\mathrm{sh}^2\,\xi + \sin^2\eta) = \gamma_k(\xi-\xi_0)^k \qquad (k = 0, 1,\ldots). \tag{5}$$

We get for the γ_k that

$$\gamma_0 = \gamma_0(\eta) = a^2(\beta-1)(\mathrm{sh}^2\,\xi_0 + \sin^2\eta),$$
$$\gamma_{2k-1} = a^2(\beta-1)\,\mathrm{sh}\,2\xi_0\frac{2^{2k-2}}{(2k-1)!} = \mathrm{const},$$
$$\gamma_{2k} = a^2(\beta-1)\,\mathrm{ch}\,2\xi_0\frac{2^{2k-1}}{(2k)!} = \mathrm{const},$$
$$(k = 1, 2, \ldots). \tag{6}$$

Since $\tanh^2\xi_0 = 1/\beta$, we get the final formulas

$$\gamma_0 = a^2H(\eta) = {}^1/_2\,a^2[(\beta+1) + (\beta-1)\cos 2\eta],$$
$$\gamma_{2k-1} = a^2\sqrt{\beta}\,\frac{2^{2k-1}}{(2k-1)!}, \qquad \gamma_{2k} = a^2(\beta+1)\frac{2^{2k-1}}{(2k)!}. \tag{7}$$

Substitution of (4) and (5) into (2) gives us recurrent relations for the φ_k:

$$s(s+1)\,\varphi_{s+1} + (\varphi_{s-1})_\eta'' + \sum_{k=0}^{s-1}\gamma_k(\varphi_{s-k-1})_z'' = 0$$
$$(s = 1, 2, \ldots). \tag{8}$$

Use of (8) gives

$$(s=1)\qquad \varphi_2 = -{}^1/_2\,\gamma_0\varphi_0'',$$
$$(s=2)\qquad \varphi_3 = -{}^1/_6\,\gamma_1\varphi_0'',$$
$$(s=3)\qquad \varphi_4 = -{}^1/_{12}\,(-{}^1/_2\,\gamma_0'' + \gamma_2)\,\varphi_0'',$$
$$(s=4)\qquad \varphi_5 = -{}^1/_{20}\,\gamma_3\varphi_0'',$$
$$(s=5)\qquad \varphi_6 = -{}^1/_{30}\,({}^1/_{24}\,\gamma_0^{\mathrm{IV}} + \gamma_4)\,\varphi_0'',$$
$$(s=6)\qquad \varphi_7 = -{}^1/_{42}\,\gamma_5\varphi_0'',$$
$$(s=7)\qquad \varphi_8 = -{}^1/_{56}\,(-{}^1/_{720}\,\gamma_0^{\mathrm{IV}} + \gamma_6)\,\varphi_0'',$$
$$(s=8)\qquad \varphi_9 = -{}^1/_{72}\,\gamma_7\varphi_0''. \tag{9}$$

In general, it is readily seen that

$$\varphi_{2k+1} = -\frac{\gamma_{2k-1}}{2k(2k+1)}\,\varphi_0'' = -a^2\sqrt{\beta}\,\frac{2^{2k-1}}{(2k+1)!}\,\varphi_0'',$$
$$(k = 1, 2, \ldots),$$
$$\varphi_{2k} = -\frac{1}{(2k-1)\cdot 2k}\left[(-1)^{k-1}\frac{\gamma_0^{(2k-2)}}{(2k-2)!} + \gamma_{2k-2}\right]\varphi_0'' =$$
$$= -\frac{2^{2k-2}}{(2k)!}\,\gamma_0(\eta)\,\varphi_0'' \quad (k = 2,3,\ldots). \tag{10}$$

We substitute (3) and (10) into (4) and sum the series to get the following expression for the potential in the region exterior to the beam:

$$\varphi = \alpha + (1-\alpha)\,Z^2 - {}^1/_2\,a^2\sigma^{-2}(1-\alpha)\,\{\sqrt{\beta}\,(\mathrm{sh}\,2\Xi - 2\Xi) +$$
$$+ [(\beta+1) - (\beta-1)\cos 2\eta]\,(\mathrm{ch}\,2\Xi - 1)\}, \qquad \Xi = \xi - \xi_0. \tag{11}$$

We use the semiminor axis of the ellipse as the characteristic linear dimension in the plane of ξ and η. With $\beta = H(\eta) = a = 1$, (11) defines the potential for a cylindrical beam, R = 1:

$$\varphi = \alpha + (1-\alpha)\,Z^2 - {}^1/_2\,\sigma^{-2}(1-\alpha)\,(R^2 - 2\ln R - 1). \tag{12}$$

This is a further form of approximate analytic solution, which differs from one previously considered [3]. Figures 2 and 3 give curves derived from intersection of the surfaces φ = const,

$$Z = \left\{\frac{\varphi-\alpha}{1-\alpha} + \frac{1}{2\sigma^2}\left[\sqrt{\beta}\,(\mathrm{sh}\,2\Xi - 2\Xi) +\right.\right.$$
$$\left.\left.+ H(\eta)\,(\mathrm{ch}\,2\Xi - 1)\right]\right\}^{1/2}, \tag{13}$$

with the half-planes ψ = const for various values of α and β. In constructing these curves it is convenient to use formulas relating ξ and η to the polar coordinates R and ψ:

$$\Xi = {}^1/_2\ln(\sqrt{\beta} + 1)^{-2}\{R^2 +$$
$$+ \sqrt{2}R\,(\cos\psi\,[\sqrt{R^4 - 2(\beta-1)R^2\cos 2\psi + (\beta-1)^2} +$$
$$+ R^2\cos 2\psi - (\beta-1)]^{1/2} + \sin\psi\,\times$$
$$\times[\sqrt{R^4 - 2(\beta-1)R^2\cos 2\psi + (\beta-1)^2} - R^2\cos 2\psi +$$
$$+ (\beta-1)]^{1/2}) + \sqrt{R^4 - 2(\beta-1)R^2\cos 2\psi + (\beta-1)^2}\},$$
$$\mathrm{tg}\,\eta = \{\sqrt{2}\,R\sin\psi +$$
$$+ [\sqrt{R^4 - 2(\beta-1)R^2\cos 2\psi + (\beta-1)^2} -$$
$$- R^2\cos 2\psi + (\beta-1)]^{1/2}\}\{\sqrt{2}\,R\cos\psi +$$

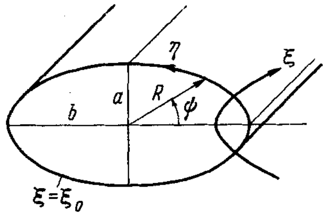

Fig. 1

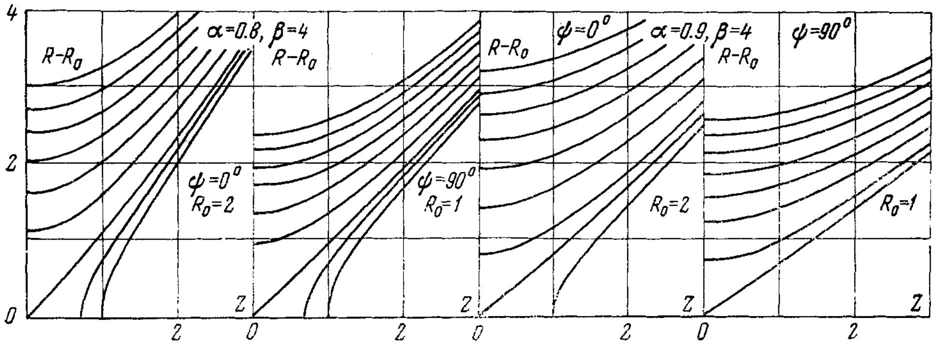

Fig. 2

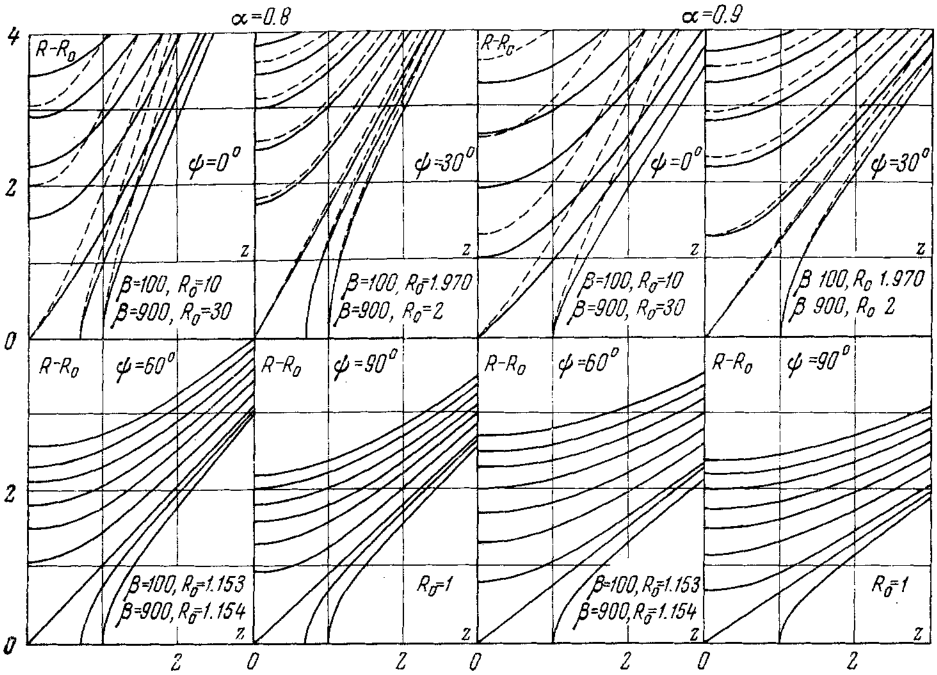

Fig. 3

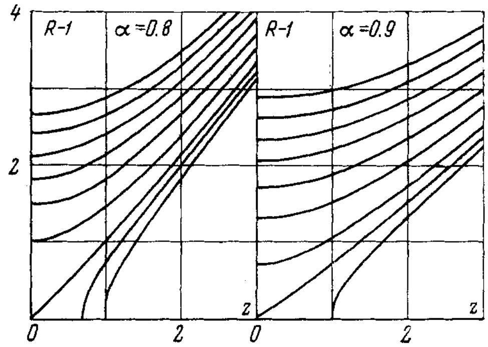

Fig. 4

$$+ [\sqrt{R^4 - 2(\beta - 1) R^2 \cos 2\psi + (\beta - 1)^2} + R^2 \cos 2\psi - (\beta - 1)]^{1/2} \}^{-1},$$

and also the equation of the boundary ellipse in these coordinates:

$$R_0 = \sqrt{\beta}\, [1 + (\beta - 1) \sin^2 \psi]^{-1/2}.$$

The curves from right to left correspond to $\varphi = 1$, 0.9, 0.8, and then with a step of 0.2. The solid lines in Fig. 3 are for $\beta = 100$, while the dashed ones are for $\beta = 900$. The difference between these two families of equipotential surfaces is small for $\psi = 30°$, while it is virtually zero for ψ of 60 and 90°.

Figure 4 shows surfaces of rotation φ = const calculated from (12). The screening grids must have a two-dimensional potential distribution for an elliptical beam; the dependence $\varphi(R, \psi)$ is easily deduced from Figs. 2 and 3.

It is to be expected that the deviation of (11) from the exact solution will be of the same order as in the planar case [1], especially where the curvature of the boundary is small.

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